A Matroid Generalization of Sperner's Lemma

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Abstract

In a 1980 paper, Lovász generalized Sperner's lemma for matroids. He claimed that a triangulation of a d-simplex labeled with elements of a matroid M must contain at least one "basis simplex". We present a counterexample to Lovász's claim when the matroid contains loops and provide a necessary condition such that Lovász's generalization holds. Furthermore, we show that under some conditions on the matroids, there is an improved lower bound on the number of basis simplices. We present further work to sharpen this lower bound by looking at M's lattice of flats and by proving that there exists a group action on the simplex labeled by M with $\mathcal{S}_n$.

1 Introduction

Sperner's lemma is a claim about the triangulations of simplices, which is noted for its equivalence to the Brouwer Fixed Point theorem. It states that given a triangulation T of a d-simplex S and a *Sperner labeling* on T, there must exist at least one *fully labeled Sperner simplex.* In [1], Lovász extends Sperner's lemma for *matroids*, a construct that generalizes the concept of linear independence. His extension states the following:

Theorem 1.1 (Lovász, 1980). *Let S be a d-simplex, K a simplicial subdivision of S and assume that a matroid of rank $d+1$ is defined on the vertices of K. Assume furthermore that the vertex-set $V(S)$ of S is independent in the matroid and that for each $A \subseteq V(S)$, those vertices of K on the face spanned by A are contained in the flat of the matroid spanned by A. Then K has a simplex whose vertices form a basis.*

This theorem asserts that there must exist at least one *basis simplex*. We found counterexamples to this theorem when the matroids contain *loops*, i.e., singleton dependent sets. We show that if we add to the hypothesis of Lovász's theorem the condition that the matroid used in the labeling is *loopless*, then the conclusion of the theorem holds.

In addition to understanding Lovasz's result, the main motivation of our project is to improve the lower bound on the number of basis simplices that we can guarantee in a matroid-labeled triangulation. That is, under what conditions on the matroids can we assure the existence of more than one basis simplex.

We give the necessary background on Sperner's lemma and matroids in Section 2. In Section 3, we formalize our corrections to Lovász's paper and provide an improved lower bound for the one dimensional case. Additionally, we provide different approaches to solve this problem in higher dimensions. Section 4 of this paper returns to Lovász's constructions and highlights a group action on the vertices of a triangulation labeled by a matroid. Finally, Section 5 is dedicated to remarks, conjectures, and future work.

2 Background

At the heart of our question is Sperner's lemma and a number of constructs from matroid theory. In this section we will define and discuss the necessary notions. Furthermore, we introduce Lovász's results that bridges these concepts, provide a correction to his paper, and prove this new claim.

2.1 Sperner's Lemma

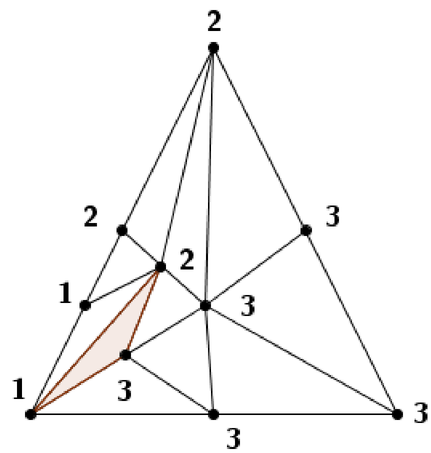

Figure 1: A Sperner labeling on a 2-simplex

We start our introduction of Sperner's lemma with a motivating example. Consider the example shown in Figure 1. For this triangulation, we start by labeling the three main vertices of the triangle distinctly by $1, 2, 3$. Then, for any vertices on an edge of the main triangle we impose the label of one of the vertices at the endpoints of the edge. For example, on the edge labeled by $1, 2$ in the main triangle we have the two vertices in between labeled arbitrarily by either 1 or 2. The vertex in between the edge labeled by 2 and 3 on the main triangle is labeled by 3 and the vertex on the edge $1, 3$ is labeled by 3; although 1 is a valid labeling. Any vertex inside the main triangle can be labeled by any element in $\{1, 2, 3\}$.

This type of labeling is what is known as a Sperner labeling for a 2-simplex. What Sperner's lemma asserts is that we have an odd number of fully labeled simplices and that there exists at least one. Fully labeled triangles are triangles labeled distinctly by elements in $\{1, 2, \ldots, d+1\}$ for a d-simplex; in this case, a triangle labeled by $1, 2$ and 3. Going back to Figure 1, the shaded simplex is the only fully labeled Sperner simplex in this triangulation.

In general, a Sperner labeling on a d-dimensional simplex S with a triangulation T is a labeling that satisfies the following rules:

- The vertices of the main simplex S are distinctly labeled by all the elements in $\{1, 2, \ldots, d+1\}$.
- The vertices located on any k-dimensional face $\{a_1, a_2, \ldots, a_{k+1}\}$ of the main simplex are labeled by any element in $\{a_1, a_2, \ldots, a_{k+1}\}$.

Then, Sperner's lemma states the following:

Lemma 2.1 (Sperner's Lemma)**.** *Any Sperner-labeled triangulation of a d-simplex must contain an odd number of fully labeled elementary d-simplices. In particular, there is at least one.*

2.2 Matroid Theory

Matroids are mathematical objects that capture the notion of linear independence in vector spaces. We follow Oxley's [2] definition and notations for our introduction to matroids.

Definition 2.2. A matroid is a pair $M = (E, \mathcal{I})$ consisting of a finite set E called the ground set and a collection of subsets $\mathcal{I}$ from E that satisfy the following conditions:

- $\emptyset \in \mathcal{I}$
- If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$
- If I_1 and I_2 are in $\mathcal{I}$, and $|I_1| < |I_2|$ then there is an element $e \in I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

It is useful to consider some examples. In the first example below, we have a matroid of vectors in $\mathbb{R}^2$ where the independent sets are sets of linearly independent vectors. The second example involves a *graphic matroid*, a type of matroids which we will use in further examples throughout the paper.

Example 1. Consider the following matrix whose columns are vectors in $\mathbb{R}^2$:

$$\begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Let $E = \{e_1, \dots, e_5\}$ denote the set consisting of the five column vectors of the above matrix and let $\mathcal{I}$ denote the collection of all subsets of E which forms linearly independent sets of the vectors in $\mathbb{R}^2$.

Then, it is not hard to see that $M = (E, \mathcal{I})$ is a matroid. The empty set of vectors is defined to be linearly independent, so $\emptyset$ is in $\mathcal{I}$. For $I \in \mathcal{I}$, a set of linearly independent vectors, then any subset $I' \subseteq I$ is also a linearly independent set of vectors, so I' must be an element of $\mathcal{I}$ as well. We leave it to the reader to verify that the third condition in Definition 2.2 also holds for M.

Example 2. Consider the graph in Figure 2 below. Let E denote the set of edges in the graph and let $\mathcal{I}$ denote the collection of all sets of edges that do not form a cycle. That is, a set $I \subseteq E$ of edges is an independent set if it does not form a cycle. Otherwise, if $X \subseteq E$ forms a cycle, we say that it is a dependent set.

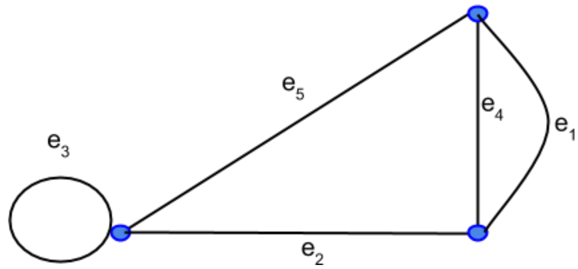

Figure 2: Example of a graphic matroid.

We claim that $M = (E, \mathcal{I})$ is a matroid. The empty set of vectors does not form a cycle, so $\emptyset$ is in $\mathcal{I}$. If I is a set of edges that do not form a cycle, then any subset $I' \subseteq I$ must not form a cycle, which means that $I' \in \mathcal{I}$. Again, we leave it to the reader to verify that the third condition also holds for M, thereby showing that M is a matroid. Such a matroid, whose ground set is the set of edges in a given graph and whose independent sets are the sets of edges with no cycles, is called a *graphic matroid*.

A minimal dependent set in an arbitrary matroid M will be called a *circuit* of M and we shall denote the set of circuits of M by $\mathcal{C}$. If a two-element set $\{m_1, m_2\}$ form a circuit in M, then m_1 and m_2 are *parallel* in M. The *parallel class* of an element $m \in E$ is then the set of all elements in E that are parallel to m.

We know from linear algebra that any set of n linearly independent vectors in $\mathbb{R}^n$ will span all of $\mathbb{R}^n$ and we call this set a *basis* of $\mathbb{R}^n$. Another useful concept is the *rank* of a set of vectors. We know that any basis in $\mathbb{R}^n$ will be of rank n, we also know that adding any other vector to a basis will make the set dependent but the rank will remain the same. This suggests a generalization of basis and rank for matroids:

- A basis of a matroid $M = (E, \mathcal{I})$ is a maximal independent set of E.
- The rank $r(M)$ of a matroid M is the size of a basis in M. Then, the rank of a subset $X \subseteq E$ is the size of the largest independent set in X.

Formally, let $M = (E, \mathcal{I})$ be a matroid, suppose that $X \subseteq E$ and that $I|X = \{I \subseteq X : I \in \mathcal{I}\}$. We define the *rank* $r(X)$ of X to be the size of a basis B of $M|X$. That is, a function $r : 2^E \to \mathbb{Z}^+ \cup 0$ is the rank function of a matroid on E if and only if r satisfies the following conditions:

- If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.
- If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.
- If X and Y are subsets of E, then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

Two concepts of utmost importance in our paper are the ideas of closures and flats. Let cl be the function from 2^E into 2^E defined for all subsets $X \subseteq E$ by

$$\mathrm{cl}(X) = \{x \in E : r(X \cup x) = r(X)\}.$$

This function is called the *closure operator* of M. A set $X \subseteq E$ is called *flat* if $X = \mathrm{cl}(X)$. Throughout this paper we will denote the flat of any set $X \subseteq E$ as $\langle X \rangle$.

2.3 Matroid and Induced Sperner's Labelings

In his paper, Lovász described a method of labeling a triangulation by elements of a matroid in a way that emulates Sperner labeling. In the case of a triangle's triangulation, we label the main vertices of said triangle with the elements of a basis of a rank 3 matroid. Then we use the flats of the basis elements labeling the main triangle to label vertices on the edges they define. Finally we label internal vertices with any remaining elements. We see this in Figure 3.

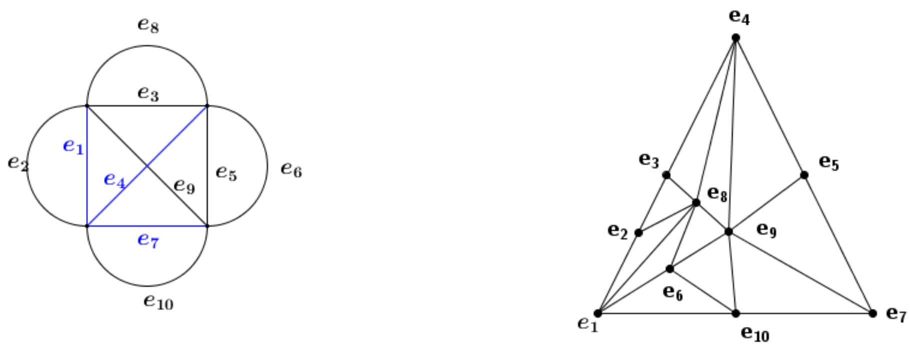

Figure 3: A matroid and its labeling on a triangulation.

In general, a labeling of this kind is defined as follows:

Definition 2.3. Let S be a d-simplex and T a triangulation of S. Assume that there exists a matroid of rank $d+1$ defined on the vertices of T. We define a *matroid labeling* on T as a labeling on the vertices of T that satisfy the following conditions:

- The vertex-set $V(S)$ of S forms a basis in the matroid.
- For each $A \subseteq V(S)$, the vertices of T that are in the face spanned by A are contained in the flat of the matroid spanned by A.

Any simplex in the triangulation whose vertices are labeled by a basis is a *basis simplex*. It is important to note that for an arbitrary matroid and triangulation there is not necessarily a proper matroid labeling. It is vital that the triangulation have the same number of vertices as the matroid has elements and that the flats of the elements labeling the main edges have enough elements to label the vertices on those edges.

Due to its similarity to Sperner labeling, matroid labeling can be converted to Sperner labeling with relative ease. We simply look at the flats of the basis elements labeling the main vertices of our simplex and impose a labeling from this. A Sperner labeling brought on from a matroid labeling is said to be induced by it.

Definition 2.4. Suppose S is a d-dimensional simplex and T corresponds to a triangulation of S. Let $M = (E, \mathcal{I})$ be a rank $(d+1)$-matroid with basis $\{a_1, ..., a_{d+1}\}$ and $F_i = \langle a_1, \dots a_i\rangle - \langle a_1 \dots a_{i-1}\rangle$ for all $i \in [d+1]$. A Sperner labeling on T induced by a matroid M is a labeling that satisfies the following properties:

- There is a matroid labeling on T.
- For all $v \in T$, v is labeled by some i corresponding to the flat F_i of v.

To highlight this process we shall look at the matroid and 2−simplex from earlier. Consider Figure 4.

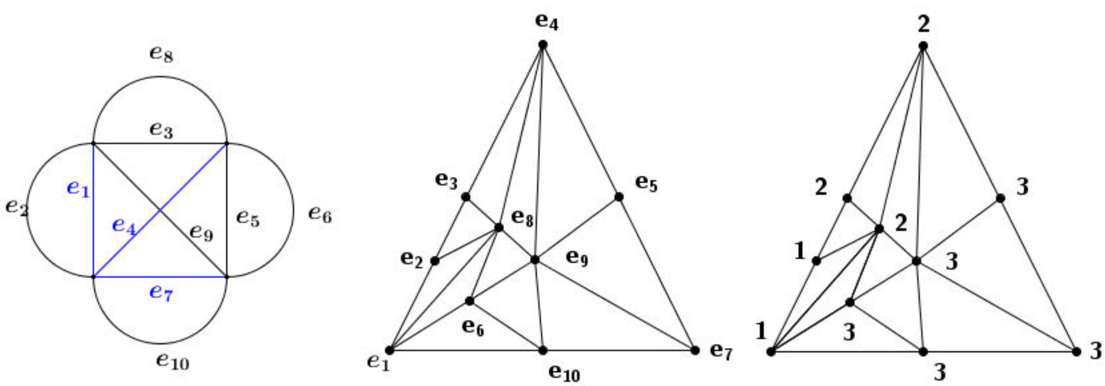

Figure 4: A matroid, its labeling on a triangulation, and its induced Sperner labeling.

Notice the vertices corresponding to the elements in the flat of e_1 are labeled with 1 in the induced Sperner labeling—namely e_1 and e_2. Then the vertices with the unlabeled elements in the flat of e_1 and e_4 are labeled with 2—namely e_3, e_4, and e_8. Lastly all of the remaining vertices are labeled with 3's in the induced Sperner labeling.

3 Results

3.1 Correction to Lovász's Results

The motivation for this paper is a result published by Lovász [1] in which he asserts the following:

Let K be a simplicial complex which is a d-dimensional manifold. Also assume that a matroid of rank $d+1$ is defined on the set of vertices of K. If K has a simplex whose vertices form a basis of the matroid, then it has at least two.

In the proof, Lovász uses the following procedure:

Assume that $(a_1, \ldots, a_{d+1})$ is the unique simplex which is a basis. Let F_i, denote the flat spanned by $\{a_1, \ldots, a_i\}$. Let Q denote the set of all sequences $(x_1, \ldots, x_d)$ of elements of the matroid such that

$$x_1 \in F_1, \qquad x_i \in F_i - F_{i-1} \tag{1}$$

Then, Lovász claims the set $\{x_1, \ldots, x_d\}$ is automatically independent in the matroid. But this is not necessarily the case.

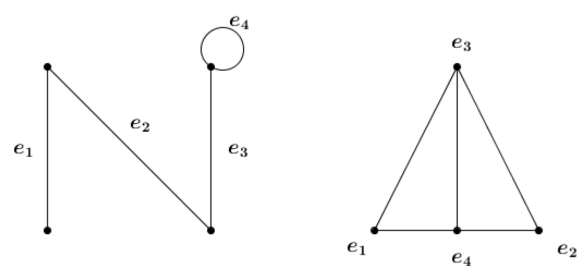

Figure 5: Lovász claims that any matroid labeling must have at least one basis simplex. Loops, which are dependent on their own, make this untrue.

Consider figure 5, in this figure the edge e_4 is a loop and is therefore dependent to itself and every other element in the matroid. Which means that in the construction in 1 every set in Q will contain a loop and will therefore be dependent. This contradicts the claim and suggests that we make the following addendum to the statement:

Theorem 3.1. *Let K be a simplicial complex which is a d-dimensional manifold. Also assume that a* ***loopless*** *matroid of rank $d+1$ is defined on the set of vertices of K. If K has a simplex whose vertices form a basis of the matroid, then it has at least two.*

We added the condition that our choices of matroids are *loopless matroids*. Loopless matroids are matroids that do not contain dependent singletons in the ground set. By restricting our choice of matroids to loopless matroids, we can be certain that the construction in 1 will yield a set Q of independent subsets of E.

Next we state the corollary presented in [1]:

Let S be a d-simplex, K a simplicial subdivision of S and assume that a matroid of rank $d+1$ is defined on the vertices of K. Assume furthermore that the vertex-set $V(S)$ of S is independent in the matroid and that for each $A \subseteq V(S)$, those vertices of K on the face spanned by A are contained in the flat of the matroid spanned by A. Then K has a simplex whose vertices form a basis.

The triangulation in figure 5 satisfies the conditions mentioned in the foregoing corollary but leads to an erroneous conclusion. There is no basis simplex in the triangulation and this is due to the fact that there is a loop in the matroid. If we then restrict our choices to loopless matroids we can prove the corollary holds.

Corollary 3.2. *Let S be a d-simplex, K a simplicial subdivision of S and assume that a **loopless** matroid of rank $d+1$ is defined on the vertices of K. Assume furthermore that the vertex-set $V(S)$ of S is independent in the matroid and that for each $A \subseteq V(S)$, those vertices of K on the face spanned by A are contained in the flat of the matroid spanned by A. Then K has a simplex whose vertices form a basis.*

To prove this corollary we first prove the following lemma:

Lemma 3.3. *Let S be a d-simplex, T a triangulation of S and $\mathcal{P}(T)$ the Sperner labeling induced by a matroid $M = (E, \mathcal{I})$ of rank $d+1$ defined on the vertices of T. If $\{v_1, v_2, \ldots, v_{d+1}\}$ are the vertices of a fully labeled Sperner simplex on $\mathcal{P}(T)$ then $\{v_1, v_2, \ldots, v_{d+1}\}$ corresponds to a basis $\{b_1, b_2, \ldots, b_{d+1}\}$ on M.*

Proof. Suppose $\{v_1, v_2, \ldots, v_{d+1}\}$ are vertices that form a fully labeled Sperner simplex on $\mathcal{P}(T)$. Let F_i denote the flat that indexes the vertex $w_i \in T$ and let $\mathcal{P}(b_i)$ denote the element in E that is labeled by $v_i \in \{v_1, v_2, \ldots, v_{d+1}\}$. Then v_i is labeled by an element b_i such that $b_i \in F_i - (F_{i-1} \cup F_{i-2} \cup \ldots \cup F_1)$, that is, b_i is independent to any element in $F_{i-1}, F_{i-2}, \ldots, F_1$. Since each element in $\{v_1, v_2, \ldots, v_{d+1}\}$ is labeled differently, then b_i is independent to any b_k such that $\mathcal{P}(b_k) = v_k \in \{v_1, v_2, \ldots, v_{d+1}\}$. This implies that the set $\{b_1, b_2, \ldots, b_{d+1}\}$ is an independent set of size $d+1$. Therefore, the set $\{v_1, v_2, \ldots, v_{d+1}\}$ corresponds to a basis $\{b_1, b_2, \ldots, b_{d+1}\}$ in M. □

Proof of Corollary 3.2. By lemma 3.3 and Sperner's lemma, the corollary follows. □

3.2 Lower Bound on Basis Simplices

By virtue of the conditions on labeling a triangulated simplex with a matroid, certain elements are limited on what vertices they can label. Due to this and their relative independence from the ordered basis we can see that certain matroids demand a minimum number of basis simplices. We will first explore this in the one dimensional case.

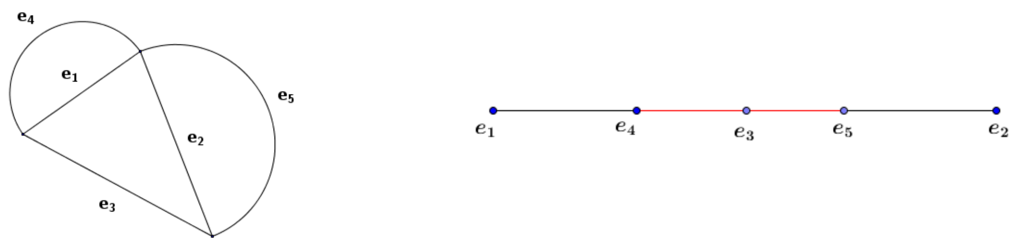

Figure 6: One dimensional example of a matroid labeling.

Consider the matroid and its corresponding triangulated $1-$simplex in Figure 6. By construction, the vertex labeled by the element e_3 can be swapped with the label of any other internal vertex on the $1-$simplex. It is easy to check that e_3 is dependent to both e_1 and e_2 but independent to either of those elements (and their parallel elements) individually. Therefore e_3 forms a basis with any element in the parallel classes of e_1 or e_2. Additionally, any element in the parallel class of e_1 will form a basis with any element in the parallel class of e_2 and vice versa. Thus, regardless of how we scramble the labels of the internal vertices on this $1-$simplex there will be at least two basis simplices. We will now generalize this example.

Theorem 3.4. *Let M be a matroid of rank 2 that has a circuit of size 3 and let S be a 1-simplex. Then, for any triangulation T of S that is labeled by M there are at least two basis simplices.*

Proof. Without loss of generality we shall refer to elements in the parallel classes of the basis elements that label the main vertices of T as P_1 and P_2 and to P_3 as any element that forms a circuit of size 3 with elements of P_1 and P_2. We now have four cases: either an element of P_3 has a fellow P_3 element and either a P_1 or P_2 element adjacent to it, that element has only P_3 elements adjacent to it, both an element in P_1 and P_2 are adjacent to it, or that element has only elements from P_1 or P_2 adjacent to it. In the first two cases we simply note that when an element of P_3 has a fellow P_3 element adjacent to it that adjacent element must fall within the cases as well. Our triangulation is finite, so it follows that either a P_1 or P_2 element must eventually be adjacent to one of these adjacent P_3 elements. As such, we can treat these chains of P_3 elements as if they were a single element and we fall into the remaining two cases.

Case 1: Suppose the element(s) in P_3 are surrounded on both sides by elements in P_1 *and* P_2. As mentioned, said element(s) form a basis with both the elements in P_1 and P_2. Thus there are at least two basis simplices.

Case 2: By a nearly identical argument, suppose the element(s) in P_3 are surrounded on both sides by elements in either P_1 *or* P_2. Regardless of which parallel class they are adjacent to, elements in P_3 form a basis with elements in either P_1 or P_2. Therefore there are at least two basis simplices again. □

Should a matroid of rank 2 without a circuit of size 3 be used in labeling a triangulation it simply falls into a case of Sperner's lemma for $1-$simplices.

When moving to higher dimensional simplices we need to be weary of overwhelmingly large parallel classes that "smother" our triangulation. To see what this means consider Figure 7.

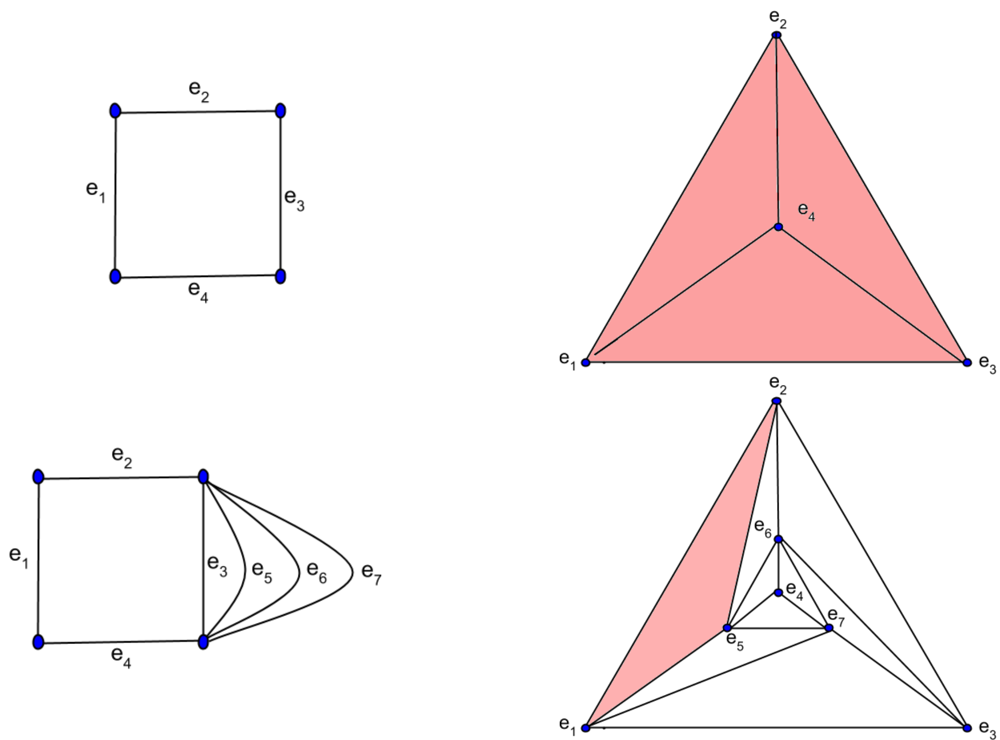

Figure 7: A matroid with a large parallel class and a "smothered" triangulation.

The huge parallel class of e_3 allows us to surround elements that would otherwise form multiple basis simplices and limit the amount of basis triangles that appear. While dealing with general matroids we have to worry about having parallel classes that run rampant.

4 Further Results

4.1 Lattice of Flats

As it was shown in the previous section, the Sperner labeling induced by a matroid M depends on the order of the basis we are fixing. This suggests that there should be an action of the symmetric group on the elements of the basis, and so, an action on the Sperner labeling. To show this, we need an auxiliary structure from algebraic combinatorics:

Definition 4.1. A *poset* P is a finite set, also denoted P, together with a binary relation denoted $\leq$ satisfying the following axioms:

- (reflexivity) $x \leq x$ for all $x \in P$.

- (antisymmetry) If $x \leq y$ and $y \leq x$, then $x = y$.
- (transitivity) If $x \leq y$ and $y \leq z$, then $x \leq z$.

For our purpose we will be interested in a particular poset:

Definition 4.2. Let $n \in \mathbb{N}$ and $E = \{1, 2, ..., n\}$ be a set. We call $\mathfrak{B}_n = (\mathcal{P}(E), \subset)$ the *boolean algebra* of n elements, where 2^E denotes the power set of E.

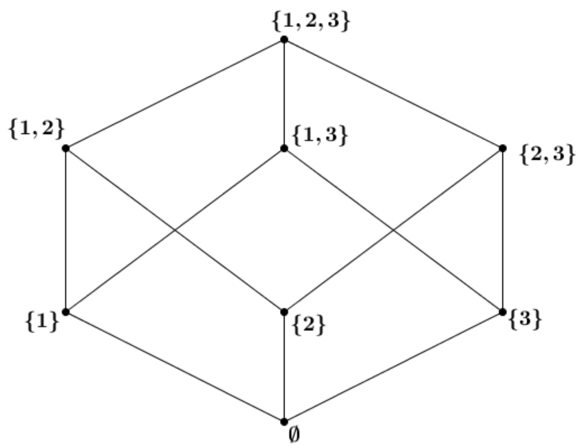

Figure 8: $\mathfrak{B}_3$.

In Figure 8 we have what is called the *Hasse-diagram* of $\mathfrak{B}_3$. Now let $\mathcal{S}_n$ be the symmetric group on n elements. For any $\sigma \in \mathcal{S}_n$ and $A = \{i_1, ..., i_m\} \in \mathfrak{B}_n$ we define $\sigma A = \{\sigma(i_1), ...\sigma(i_m)\}$ where $m \leq n$. It's easy to check that this is indeed an action of $\mathcal{S}_n$ on $\mathfrak{B}_n$. From now on we are going to drop the brackets and commas when we talk about elements of $\mathfrak{B}_n$, for example: $\{1, 2, 3\}$ will be denoted by 123.

A *lattice* is a poset $\mathcal{L}$ such that, for every pair of elements, the least upper bound and the greatest lower bound of the pair exists. Formally, if x and y are arbitrary elements of $\mathcal{L}$, then $\mathcal{L}$ contains elements $x \vee y$ and $x \wedge y$, the *join* and *meet* of x and y respectively, such that:

- $x \vee y \geq x, x \vee y \geq y,$ and if $z \geq y,$ then $z \geq x \vee y$; and
- $x \wedge y \leq x, x \wedge y \leq y,$ and if $z \leq y,$ then $z \leq x \wedge y$.

In the case of $\mathfrak{B}_3$ the operations join and meet are union and intersection, respectively. It is easy to see from the Hasse-diagram that $\mathfrak{B}_3$ is in fact a lattice.

If M is a matroid, let $\mathcal{L}(M)$ denote the sets of flats of M ordered by inclusion. It's easy to see that this is a partially ordered set. Additionally it can be endowed with the structure of a lattice, as stated by the following theorem.

Theorem 4.3. *$\mathcal{L}(M)$ is a lattice and, for all flats X and Y of M,*

$$X \wedge Y = X \cap Y \text{ and } X \vee Y = cl(X \cup Y).$$

4.2 Group Action

A Sperner labeling induced by a matroid labeling as described in **section 2.3**, is fully reliant on the order in which we consider the chosen basis. By simply changing the order of the basis we induce vastly differing Sperner labels with varying fully labeled elementary simplices. By the lemma presented in **section 4.2.2**, this means that the Sperner triangles may correspond to different basis simplices altogether. We will show that there exists a group action on the induced Sperner labelings as we permute the order of the basis elements.

4.2.1 $\mathcal{S}_3$ as a Group Action

Our goal for this section is to extend the action defined in **section 4.1** to an action on the Sperner labeling by permuting the order of the basis, first we do it for a given example and then generalize in the next subsection. Let M be the matroid depicted in Figure 9 and choose $B = \{e_1, e_4, e_7\}$, an ordered basis of M. Notice that in $\mathcal{L}(M)$, Figure 10, restricting ourselves to the flats that include $\langle e_1 \rangle$, $\langle e_4 \rangle$, or $\langle e_7 \rangle$ gives a sublattice P_B that looks as a boolean algebra (Figure 11). This suggests that there is an isomorphism between P_B and $\mathfrak{B}_3$, and since the basis B was ordered, the most natural isomorphism ϕ would be to send $\langle e_1 \rangle$ to 1, $\langle e_4 \rangle$ to 2, and so on.

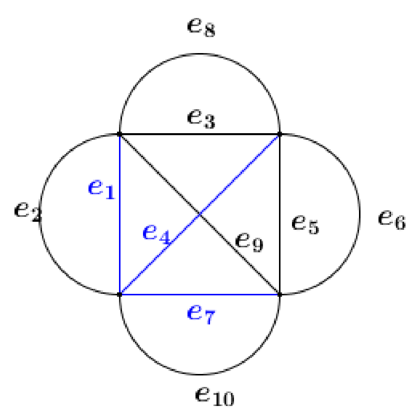

Figure 9: The matroid M.

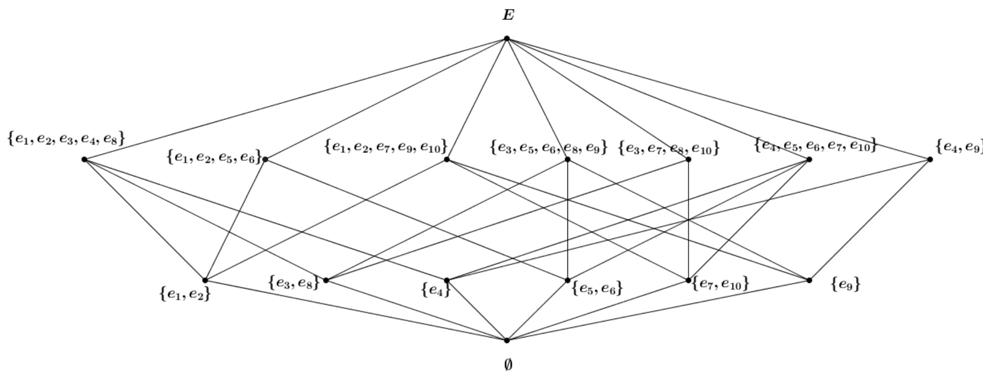

Figure 10: The lattice $\mathcal{L}(M)$ corresponding to M in Figure 9.

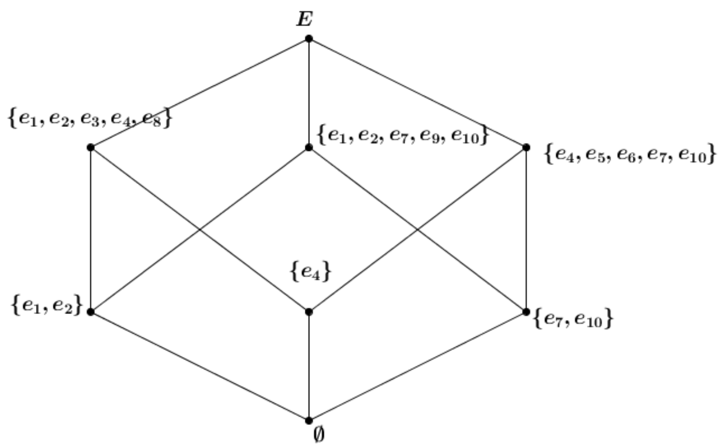

Figure 11: Restriction of $\mathcal{L}(M)$ to P_B.

Now suppose that we have a triangulation T on a triangle that is matroid-labeled by a matroid M (Figure 12). We are now going to show a different way to construct the induced Sperner labeling from B. First, take the path in P_B whose vertices are the flats for the induced Sperner labeling and label each of them by $1, 2$ and 3 in order of appearance from bottom to top(Figure 13). For any other nonempty element X in P_B, we label it so that it has the same label of the first element on the path that contains it. If we label each $x \in M$ with the label of the first set that contains x we will get an Sperner labeling on T.

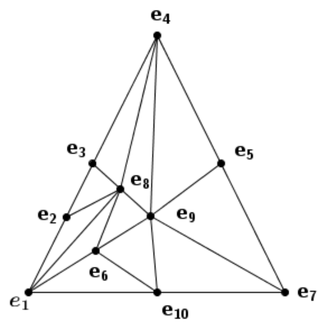

Figure 12: A triangle triangulated and matroid-labeled by M.

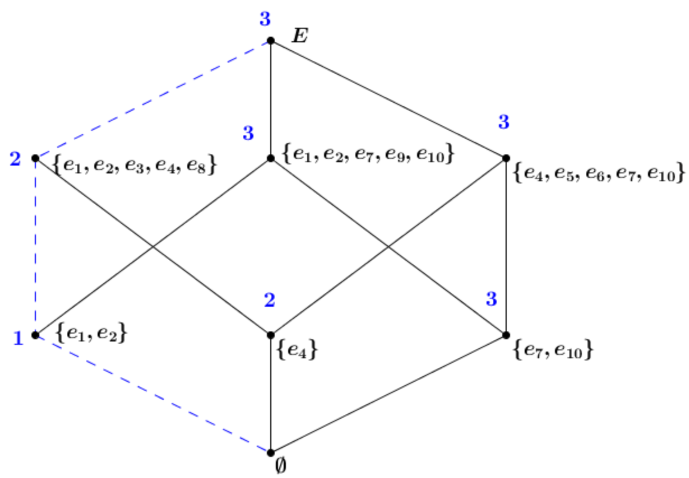

Figure 13: Sperner labeling in terms of P_B.

A natural question arises; for any order of the basis, can its respective induced Sperner labeling be constructed like this? It is indeed the case and we will give an example in Figure 14 that shows the induced Sperner labeling for the basis $B' = \{e_4, e_7, e_1\}$. This can be viewed as the basis B permuted by (132).

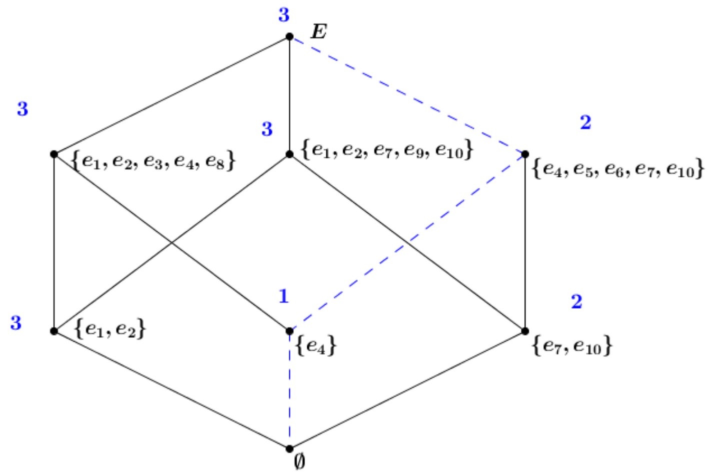

Figure 14: Sperner labeling for basis B'.

Notice that the given Sperner labeling induced by B can be assigned to the boolean algebra $\mathfrak{B}_3$ through the isomorphism ϕ. Now, if we let $\mathcal{R}$ be the set of pairs (A, b) where $A \neq \emptyset \in \mathfrak{B}_3$ and b is the respective Sperner labeling of A, we can extend the action of $\mathcal{S}_3$ on $\mathfrak{B}_3$ to $\mathcal{R}$ by letting $\sigma(A, b) = (\sigma(A), c)$ where $\sigma \in \mathcal{S}_3$ and c is the respective Sperner labeling of $\sigma(A)$. Notice that this is well defined because every element of $\mathfrak{B}_3$ is paired uniquely to one of the elements of $\{1, 2, 3\}$. In Figure 15, we show all the different pairs that live in $\mathcal{R}$.

Element of **Bn**	Sperner labelling
1	1
2	2
3	3
12	2
13	3
23	3
123	3

Figure 15: $\mathcal{R}$.

So far the action acts in some auxiliary set and it is not so clear how permuting objects in the basis will relate to those tuples in $\mathcal{R}$. Since P_B is isomorphic to $\mathfrak{B}_3$ we can think of each flat of P_B as an element of $\mathfrak{B}_3$, and since every element in the matroid appears in some element of the lattice P_B we can "label" each element of M with the tuple in $\mathcal{R}$ corresponding to the first "flat" in $\mathfrak{B}_3$ that contains it.

As an example consider $e_3 \in M$, its label would be $(12, 2)$ since the first flat that contains e_3 is $\{e_1, e_2, e_3, e_4, e_8\}$ which corresponds to 12 in the isomorphism. And, if we were going to change the order of the basis by permuting it with (132), the new labeling of e_3 would be $(132)(12,2) = (13,3)$ since $(123)(\{1,2\}) = \{1,3\}$ and 13 is paired with 3. Although the action is defined on the labelings of the elements of M it is really an action on the elements of the basis B, since the elements of B are those who are labeled by tuples of the form (i,i) with $i \in \{1,2,3\}$. To see that the action indeed corresponds to changing the order of the basis and then finding the induced Sperner labeling the reader should note that the action is permuting maximal paths on the boolean algebra. As an example of how the action would look in a triangulation we give Figure 16.

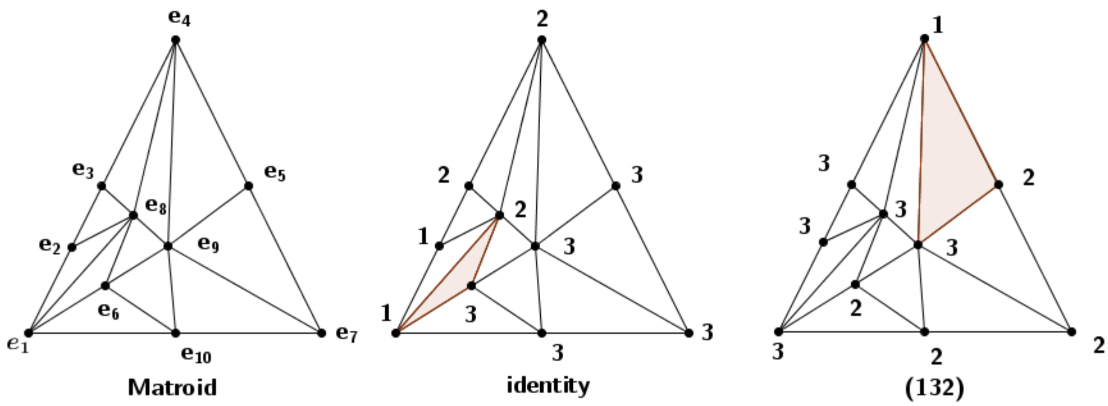

Figure 16: Example of a triangulation labeled by M, the induced Sperner labeling and by permuting the elements of the basis by (132).

4.2.2 $\mathcal{S}_n$ as a Group Action

We will now generalize this concept to any dimension. Let M be a matroid on the vertices of a $(d-1)$-simplex T such that M induces a matroid labeling. Let the ordered basis corresponding to the main vertices of T be $B = \{b_1, ..., b_d\}$. For $H_i = \langle b_i \rangle$ with $i \in [d]$, we denote P_B as the poset generated by the H_i under the $\vee$ and $\wedge$ operations. We call P_B the *boolean algebra* induced by B.

A map f such that $f(H_i) = \{i\}$ easily constructs a poset $\mathfrak{B}_d$. An isomorphism between the posets P_B and $\mathfrak{B}_d$ follows trivially.

The reader should note that inducing a Sperner labeling from B', an ordered basis using the elements of B, is equivalent to:

- Taking a "maximal" path[1], $w = w_1...w_{d+1}$, in P_B and —beginning at level 1 then moving up —labeling $f(w_i)$ and its unlabeled subsets in $\mathfrak{B}_d$ as i.
- Label each element x of M with the label of $f(A)$, where A is the smallest set in P_B that contains x.

[1]We mean maximal without the empty set.

Note that by the previous method each element of $\mathfrak{B}_d$ is uniquely represented by a Sperner label. Hence, every Sperner labeling induced by B' is uniquely determined by a path in P_B (we are taking the path so that we may label). Moreover, when we take $B' = B$, because our construction of the isomorphism depends on the order the basis, the path that corresponds to B is the leftmost path in $\mathfrak{B}_d$.

When fixing an ordered basis B, define $\mathcal{R}_B$ as the set of tuples $(A, l(A))$ where $A \in \mathfrak{B}_d$ and $l(A)$ is the unique Sperner label attached to A as described above. Given $\sigma \in \mathcal{S}_d$, the action on $\mathfrak{B}_d$ induces an action on $\mathcal{R}_B$ where $\sigma(A, l(A)) = (\sigma(A), l(\sigma(A))$. Here $l(\sigma(A))$ represents the Sperner label that accompanies $\sigma(A)$. Formally stated we have the following:

Theorem 4.4. *Suppose T is a triangulation of a $(d-1)$-simplex that is matroid-labeled by M and let P_B be the boolean algebra induced by B, the ordered basis labeling the main vertices of T. The action on $\mathcal{R}_B$ corresponds to an action on the basis B and, therefore, an action in the uniquely determined induced Sperner labelings.*

Proof. To show that the action described before is in fact an action on the induced Sperner labelings we need to show two things:

- That the action on $\mathcal{R}_B$ is in fact an action.
- The action permutes paths in P_B while permuting the elements on the basis.

First we will show that the action is indeed an action: Suppose that $A \in \mathfrak{B}_d$ and $l(A)$ is the Sperner label such that $(A, l(A)) \in \mathcal{R}_B$. It follows that for the identity, $e \in \mathcal{S}_d$, $e(A, l(A)) = (e(A), l(e(A))) = (A, l(A))$. Furthermore, if $\sigma, \tau \in \mathcal{S}_d$ we have that:

$$\sigma(\tau(A, l(A))) = \sigma(\tau(A), l(\tau(A))) = (\sigma\tau(A), l(\sigma\tau(A))).$$

Then it follows that $\mathcal{S}_d$ acts on $\mathcal{R}_B$.

Now we will show that the action permutes paths in P_B while permuting the order of the basis. First, label each element $x \in M$ by $(A, l(A)) \in \mathcal{R}_B$, where $A = f(X)$ and X is the smallest set in P_B that contains x. We want to show that for a given $\sigma \in \mathcal{S}_d$ the induced Sperner labeling by B', the basis B reordered with σ, will correspond to the one of $\sigma\mathcal{R}_B$.

Let $x \in M$ and X be the flat such that X is the minimum element in P_B that contains x. By construction, x is labeled by $(f(X), i) \in \mathcal{R}_b$ for some i so that $f(X) \subseteq \{1, ..., i\}$. We know by definition of the action that $\sigma(f(X), i) = (\sigma f(X), m)$ where $m = \max\limits_{i \in f(X)} \sigma(i)$. Moreover we know that:

$$\sigma f(X) \subset \{\sigma(1), \ldots, \sigma(i)\} \subset \{1, \ldots, m\}.$$

So, when we apply σ^{-1} to the sequence of sets we will have that

$$f(X) \subset \{1, \ldots, i\} \subset \{\sigma^{-1}(1), \ldots, \sigma^{-1}(m)\}$$

because σ is an automorphism of $\mathfrak{B}_d$. Moreover, by using the fact that f is an isomorphism we have that:

$$X \subset \langle b_1, \ldots, b_i \rangle \subset \langle b_{\sigma^{-1}(1)}, \ldots, b_{\sigma^{-1}(m)} \rangle.$$

Notice that $b_{\sigma^{-1}(i)}$ is the element of the basis B that is in the "ith" position in B'. Now by the discussion before the theorem, we know that inducing a Sperner labeling by B' depends uniquely on a path $w = w_1 .... w_{d+1}$ where $w_i = \langle b'_1, ..., b'_i \rangle$ and b'_j is the element of B' in the position j. Therefore by definition: $w_m = \langle b_{\sigma^{-1}(1)}, ..., b_{\sigma^{-1}(m)} \rangle$. By construction of $\langle b_{\sigma^{-1}(1)}, ..., b_{\sigma^{-1}(m)} \rangle$ we know that w_m is the smallest element of the path that contains x. Thus, x is labeled by m in the induced Sperner labeling. We conclude that both the induced Sperner labeling and the labeling from $\mathcal{R}_B$ coincide. Hence the action on $\mathcal{R}_B$ is an action on the induced Sperner labelings.

□

With this group action we can now revisit the concepts from Section 2 in a new light. Specifically we will revisit the lemma we used to prove Lovász's corollary.

Lemma 4.5. *Suppose $\sigma \in \mathcal{S}_d$. Let S be a $d-1$-simplex, T a triangulation on S, and $\mathcal{P}(T)$ the Sperner labeling induced by a matroid $M = (E, \mathcal{I})$ of rank d defined on the vertices of T as previously described. Let $\sigma(\mathcal{P}(T))$ be the Sperner labeling induced by applying σ to the poset of the ordered basis on the main vertices of T. If there is a fully labeled Sperner triangle in $\sigma(\mathcal{P}(T))$ then its vertices correspond to a basis in M.*

Proof. This follows by construction. Suppose the ordered basis that that induced $\mathcal{P}(T)$ is $B = \{b_1, ..., b_d\}$. It follows that $\sigma(\mathcal{P}(T))$ has the ordered basis $B = \{b_{\sigma^{-1}(1)}, \ldots, b_{\sigma^{-1}(d)}\}$ and that if a vertex, x, is labeled by i then $x \in F_{\sigma^{-1}(i)} - (F_{\sigma^{-1}(i-1)} \cup ... \cup F_{\sigma^{-1}(1)})$. As such x must be independent to any element in $F_{\sigma^{-1}(i-1)} \cup ... \cup F_{\sigma^{-1}(1)}$ and has a different Sperner label than any of them. Clearly any fully labeled triangle must correspond to d independent elements. Thus, by the equicardinality of basis, a fully labeled triangle must correspond to a basis in M. □

5 Conclusion

Finding a lower bound on the number of basis simplices is heavily reliant on the triangulation being labeled and on the matroid being used to label it. We are currently working to find the necessary conditions for smothering. We believe the following to be true.

Conjecture 5.1. *Let M be a matroid such that there exists a parallel class with $r(M) + 1$ elements. Also suppose S is an $(r(M) - 1)-$simplex. Then there exists a triangulation T of S and a matroid labeling of M on T such that there is only* 1 *basis simplex.*

If we manage to prove this, we would know that finding a lowerbound different from 1 is impossible for certain matroids. With the knowledge of what matroids will not cause smothering, we can proceed with our main concern of finding a sharpened lower bound. We have some conjectures as to how this can be accomplished by looking at the lattice of flats.

If an improved lower bound is found for general d-simplices, we believe this problem can be generalized further to a version analogous to Sperner's lemma for polytopes.

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